STABILITY IN THE FIRST APPROXIMATION FOR STOCHASTIC SYSTEMS

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The first attempt to extend the Liapunov theorems on stability in the first approximation to the case of stochastic systems, was made in [1]. Special assumptions on the character of random effects were made in order to show that the complete system is stable in some probabilistic sense, if the linearized system is exponentially stable in the quadratic mean. Similar conclusion was reached in [2 and 3] for the processes defined by a system of stochastic differential equations in the sense of Ito. In [4] it was shown that for the theorem of stability in the first approximation to hold, it is sufficient that the linearized system has exponentially stable momenta of some positive degree p. The present paper shows that the moments of sufficiently small positive power are stable, whenever the linear system shows an asymptotic, almost sure stability uniform in t. Sufficient conditions of instability in the first approximation are also given.

1. Following [2 to 4], we shall investigate the system of Ito stochastic equations (see e.g. [5])

$$dX(t) = b(t, X)\,dt + \sum_{r=1}^{k} \sigma_r(t, X)\,d\xi_r(t) \tag{1.1}$$

Here $X(t)$, $b(t, x)$ and $\sigma_r(t, x)$ are vectors belonging to an l-dimensional Euclidean space E_l, while $\xi_r(t)$ are independent Wiener processes. We shall denote by $X^{s,x}(t)$ a solution of this system satisfying the initial condition $X^{s,x}(s) = x$.

In addition to (1.1), we shall consider a linear stochastic system

$$dX(t) = BX\,dt + \sum_{r=1}^{k} \sigma_r X\,d\xi_r(t) \tag{1.2}$$

and we shall assume at first, that the square, l-th order matrices $B, \sigma_1, \ldots, \sigma_r$ are constant.

Theorem 1.1. If the relation

$$P\{|X^{s,x}(t)| \to 0\ (t \to \infty)\} = 1 \tag{1.3}$$

holds for a solution of a linear stochastic system with constant coefficients (1.2), then this system is exponentially p-stable for all sufficiently small, positive values of p.

Three lemmas which follow, will yield the proof of this theorem.

Lemma 1.1. If condition (1.3) holds, then such $p > 0$ can be found, that

$$\sup_{|x|=1} M\{\sup_{t>s} |X^{s,x}(t)|^p\} < \infty \tag{1.4}$$

Proof. Let $X^j(t)$ be a solution of (1.2) determined by the initial condition $X_i^{(j)}(s) = \delta_i^j$, where δ_i^j is the Kronecker delta. Then, by the linearity of (1.2) and uniqueness of its solution, we obtain

$$X^{s,x}(t) = \sum_{i=1}^{l} x_i X_i(t)$$

where $x_1, \ldots, x_l$ are the coordinates of the vector x.

From (1.3) it follows that $P\{\max_i \sup_{t>s} |X_i(t)| < \infty\} = 1$. Therefore we have, for some constant $k > 0$,

$$P\{\max_i \sup_{t>s} |X_i(t)| < k\} \geqslant 1/2 \tag{1.5}$$

which, together with the obvious inequality

$$\sup_{t>s} |X^{s,x}(t)| \leqslant \sum_{i=1}^{l} |x_i| \sup_{t>s} |X_i(t)| \leqslant l\,|x| \max_i \sup_{t>s} |X_i(t)|$$

yields

$$\sup_{|x|<(kl)^{-1}} P\{\sup_{t>s} |X^{s,x}(t)| > 1\} \leqslant 1/2$$

Further, by the linearity of (1.2) we have

$$X^{s,\gamma x}(t) = \gamma\, X^{s,x}(t) \tag{1.6}$$

hence at any k we have, for $\alpha = \log_2(kl)$,

$$\sup_{|x|<2^{k\alpha}} P\{\sup_{t>s} |X^{s,x}(t)| > 2^{\alpha(k+1)}\} \leqslant 1/2 \tag{1.7}$$

Since the Markov process defined by (1.2) is homogeneous in time, the probability term appearing in the left-hand side of the inequality (1.7) is independent of s.

Let us denote, by $\tau^{s,x}$, the instant when the trajectory of the process reaches the set $|x| = 2^{\alpha}$ for the first time. Using the strictly Markov property (see [6]) of the process $X^{s,x}(t)$ and (1.7), we obtain the inequalities

$$\sup_{|x|\leqslant 1} P\{\sup_{t\geqslant s} |X^{s,x}(t)| > 2^{2\alpha}\} =$$

$$= \sup_{|x|\leqslant 1} \int_{u=s}^{\infty} \int_{|y|=2^{\alpha}} P\{\tau^{s,x} \in du,\ X^{s,x}(u) \in dy\}\, P\{\sup_{t\geqslant u} |X^{u,y}(t)| > 2^{2\alpha}\} \leqslant$$

$$\leqslant \frac{1}{2} \sup_{|x|\leqslant 1} P\{\tau^{s,x} < \infty\} = \frac{1}{2} \sup_{|x|\leqslant 1} P\{\sup_{t\geqslant s} |X^{s,x}(t)| > 2^{\alpha}\} \leqslant \frac{1}{2^2}$$

. . . .

$$\sup_{|x|\leqslant 1} P\{\sup_{t\geqslant s} |X^{s,x}(t)| > 2^{\alpha k}\} \leqslant \frac{1}{2^k} \tag{1.8}$$

which prove the Lemma for $p < 1/\alpha$, since when $|x| = 1$,

$$M[\sup_{t>s} |X^{s,x}(t)|^p] \leqslant \sum_{k=1}^{\infty} 2^{k\alpha p} P\{2^{(k-1)\alpha} \leqslant \sup_{t\geqslant s} |X^{s,x}(t)| < 2^{k\alpha}\} \leqslant$$

$$\leqslant \sum_{k=1}^{\infty} 2^{k\alpha p}\, 2^{-(k-1)} \leqslant 2 \sum_{k=1}^{\infty} 2^{-k(1-\alpha p)} < \infty$$

Lemma 1.2. If condition (1.3) holds then we have, for $p < 1/\alpha$,

$$\sup_{|x|=1} M |X^{s,x}(t)|^p \to 0 \qquad (t \to \infty) \tag{1.9}$$

Proof of this Lemma follows immediately from (1.3) and (1.4), together with the Lebesgue theorem (see [7]).

The Lemma which follows, is analogous to Theorem 6.1 of [1], where only the case of $p = 2$ was considered.

Lemma 1.3. If the relation (1.9) holds for a linear system with constant coefficients, then this system is exponentially p-stable in the sense of [4], i.e. for some constants A and α and for all s and x, the relation

$$M |X^{s,x}(t)|^p < A |x|^p e^{-\alpha(t-s)} \tag{1.10}$$

holds.

Proof. From Lemma 1.2 it follows that for any $Q < 1$ such τ can be found, that

$$\sup_{|x|=1} M\,|X^{s,x}(s+\tau)|^p < Q$$

Putting $Q = e^{-1}$ and taking (1.6) into account, we can write the last relation in the form

$$M\,|X^{s,x}(s+\tau)|^p \leqslant e^{-1}|x|^p \tag{1.11}$$

Further we have

$$M\,|X^{s,x}(s+2\tau)|^p = \int P\{X^{s,x}(s+\tau)\in dy\}\,M\,|X^{s+\tau,y}(s+2\tau)|^p \leqslant$$

$$\leqslant e^{-1}\int P\{X^{s,x}(s+\tau)\in dy\}\,|y|^p \leqslant e^{-2}|x|^p$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$M\,|X^{s,x}(s+k\tau)|^p \leqslant e^{-k}|x|^p \tag{1.12}$$

Let

$$t = s + n\tau + t_1 \quad (0 \leqslant t_1 < \tau), \qquad K = \sup_{t>s,|x|=1} M\,|X^{s,x}(t)|^p$$

Here $k < \infty$ by virtue of Lemma 1.1. Then, taking into account (1.12) and (1.6) we obtain

$$M\,|X^{s,x}(t)|^p = \int P\{X^{s,x}(s+n\tau)\in dy\}\,M\,|X^{s+n\tau,y}(t)|^p \leqslant$$

$$\leqslant KM\,|X^{s,x}(s+n\tau)|^p \leqslant k\,|x|^p e^{-n} \leqslant Ke\,|x|^p \exp\left\{-\frac{t-s}{\tau}\right\}$$

which proves both, Lemma 1.3 and Theorem 1.1.

Inspecting an example of a deterministic system given by $dx/dt = -x/(t+1)$ we see that the Theorem 1.1 cannot, generally, be applied to the systems with time-dependent coefficients. If however the condition (1.3) holds uniformly in some sense, then the analogous theorem is true.

Theorem 1.2. If, for solutions of the following linear stochastic system:

$$dX(t) = B(t)X\,dt + \sum_{r=1}^{k}\sigma_r(t)X\,d\xi_r(t) \tag{1.13}$$

condition

$$\sup_{s>0} P\{\sup_{u>s+T}|X^{s,x}(u)| > \varepsilon\} \to 0 \qquad (T\to\infty) \tag{1.14}$$

holds for every $\varepsilon > 0$, then this system is exponentially p-stable for all, sufficiently small and positive values of p.

The proof of this Theorem is analogous to that of Theorem 1.1.

Note. If the process is homogeneous in time, then the conditions (1.14) and (1.3) are equivalent. This follows from the equivalence of events

$$A = \{X^{s,x}(t) \nrightarrow 0\ (t\to\infty)\}, \qquad B = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\left\{\sup_{u>n}|X^{s,x}(u)| > \frac{1}{m}\right\}$$

and the resulting equality

$$P(A) = \lim_{m\to\infty}\lim_{n\to\infty} P\{\sup_{u>n}|X^{s,x}(u)| > 1/m\}$$

Therefore Theorem 1.2 is indeed a generalization of the Theorem 1.1, extending to the case inhomogeneous with respect to time.

Comparing the Theorems 1.1 and 1.2 together with the Theorem 4.1 from [4], we obtain the following assertion:

Theorem 1.3. Let the linear system (1.13) be uniformly and asymptotically stable in the sense of (1.14) and let the elements of the matrices $B(t)$ and $\sigma_r(t)$ be bounded. Then the solution $X(t) = 0$ of (1.1) is almost surely stable for all systems whose coefficients admit the validity of the estimate

$$|b(t,x) - B(t)x| + \sum_{r=1}^{k}|\sigma_r(t,x) - \sigma_r(t)x| < \gamma|x| \tag{1.15}$$

sufficiently near the point $x = 0$ and with sufficiently small constant γ.

2. Next we shall consider the following question: under what conditions the instability of the linearized system implies the instability of the complete system. We can see from the example which follows, that the situation is more complex. As it is shown in [8], a one-dimensional system

$$dX(t) = (1 - \varepsilon)\, X\, dt + \sqrt{2}\, X\, d\xi(t)$$

is stable for any $\varepsilon > 0$, and unstable when $\varepsilon = 0$. This implies that the instability of a linear stochastic system does not necessarily imply the instability of a system close to it in the sense of (1.15), no matter how small is the constant γ. We shall nevertheless show, that the analog of the Theorem 1.3 is valid when the linear system exhibits sufficiently strong instability.

We shall first define the exponential q-instability, analogous to the definition of the exponential p-stability given by (1.10).

The solution $X(t) \equiv 0$ of (1.1) shall be called exponentially q-unstable ($q > 0$) if the condition

$$M\,|X^{s,x}(t)|^{-q} < A\,|x|^{-q}\, e^{-\alpha(t-s)} \tag{2.1}$$

holds for some A and $\alpha > 0$.

Theorem 2.1. The solution $X(t) \equiv 0$ of (1.13) is exponentially q-unstable if and only if a function $V(t, x)$ homogeneous in x and of the order $-q$ exists, which satisfies the conditions ($k_i > 0$):

$$k_1\,|x|^{-q} \leqslant V(t, x) \leqslant k_2\,|x|^{-q}$$

$$L_0V = \frac{\partial V}{\partial t} + (B(t)x, \nabla)V + \frac{1}{2}\sum_{r=1}^{k} (\sigma_r(t)x, \nabla)^2 V \leqslant -k_3\,|x|^{-q} \tag{2.2}$$

$$\left|\frac{\partial v}{\partial x_i}\right| \leqslant k_4\,|x|^{-g-1}, \qquad \left|\frac{\partial^2 V}{\partial x_i \partial x_j}\right| \leqslant k_4\,|x|^{-q-2}$$

where ∇, as usual, denotes a vector with coordinates $\partial/\partial x_i$. Proof of this Theorem is analogous to that of the Theorem 4.1 in [5].

Theorem 2.2. If the solution $X(t) \equiv 0$ of (1.13) with bounded coefficients is exponentially q-unstable, then it is almost surely unstable for all systems of the form (1.1) the coefficients of which admit, in sufficiently small neighborhood of the point $x = 0$, estimates (1.15) with sufficiently small γ.

Proof. Let V satisfy (2.2). As we know, the differential generator of the system (1.1) has the form

$$L = \frac{\partial}{\partial t} + (b(t, x), \nabla) + \frac{1}{2}\sum_{r=1}^{k} (\sigma_r(t, x), \nabla)^2$$

which, together with (1.15) and (2.2) yields in sufficiently small neighborhood of $x = 0$ the estimate

$$LV = L_0V + (b(t, x) - B(t)x, \nabla)V + \frac{1}{2}\sum_{r=1}^{k} (\sigma_r(t, x) - \sigma_r(t)x, \nabla)(\sigma_r(t, x) + \\ + \sigma_r(t)x, \nabla)V \leqslant -k_3\,|x|^{-q} + \gamma\,|x|\,k_4\,|x|^{-q-1} + \gamma c\,|x|^{-q} \tag{2.3}$$

where the constant c depends on k_4 and on the upper bound of the moduli of the coefficients of (1.13). From (2.3) it follows that with suitable choice of $\gamma > 0$, the estimate $LV < 0$ when $|x| < \varepsilon$.

Since, in addition, $\inf_{t>0} V(t, x) \to \infty$ as $x \to 0$ we obtain, applying the Theorem 2.3 from [8], the required proof (Theorem 2.3 from [8] is proved for the case homogeneous in time and nondegenerate diffusion. This limitation can however be easily removed by using in the course of the proof, the methods of the theory of stochastic differential equations, see e.g. [3 and 4]).

Note: We have remarked in [8] that the existence of a Liapunov function possessing the properties given in this paper, can lead to a stronger assertion than that of negating the almost sure stability. We can in fact say that, when the condition of Theorem 2.2 holds for any $s > 0$ and x, the event $\{X^{s,x}(t) \to 0 \text{ as } t \to \infty\}$ has zero probability.

Inspecting Theorem 2.1 we easily see that the deterministic linear system with constant coefficients is q-unstable if the real parts of all the roots of the characteristic equation are positive, i.e. if the modulus of any solution except the trivial $x \equiv 0$ tends to infinity as $t \to \infty$. An analogous statement is true for stochastic systems.

Theorem 2.3. If the relation

$$P\{\,|X^{s,x}(t)| \to \infty\,(t \to \infty)\} = 1 \tag{2.4}$$

holds for solutions of the linear system (1.2) with constant coefficients when $x \neq 0$, then this system is exponentially q-unstable for all sufficiently small $q > 0$. This statement is also true for the systems with variable coefficients (1.13), if the condition (2.4) is replaced by the condion:

$$\sup_{s>0} P\{\inf_{u>s+T} |X^{s,x}(u)| < A\} \to 0 \quad \text{when } T \to \infty \tag{2.5}$$

for any $A > 0$ and $x \neq 0$.

Proof of this Theorem is almost identical to those of Theorems 1.1 and 1.2.

From Theorems 2.2 and 2.3 we obtain:

Theorem 2.4. If the solutions of the linear system (1.13) (of the system (1.2)) satisfy the condition (2.5) (the condition (2.4)) and elements of the matrices $B, \sigma_1, \dots, \sigma_k$ are bounded, then the solution $X(t) \equiv 0$ of (1.1) is almost surely unstable for all systems whose coefficients allow, in a sufficiently small neighborhood of the point $x = 0$, the estimate (1.15) with sufficiently small γ.

3. The conditions of validity of the Theorem on instability in the first approximation obtained in Section 2, can be improved. Let us consider, in particular, a one-dimensional system

$$dX(t) = b(t, X)\,dt + \sigma(t, X)\,d\,\xi(t) \tag{3.1}$$

for which the corresponding linearized system

$$dX(t) = b_0 X\,dt + \sigma_0 X\,d\,\xi(t)$$

has constant coefficients. When $b_0 < \sigma_0^2/2$ we can apply Theorem 1.3, while when $b_0 > \sigma_0^2/2$, we can apply Theorem 2.4. When $b_0 = \sigma_0^2/2$, the linear system is unstable but it is not asymptotically q-unstable for any $q > 0$. As we said before, in the latter case Theorem 2.4 on instability in the first approximation breaks down. If, however, we require that the differences $b(t, x) - b_0 x$ and $\sigma(t, x) - \sigma_0 x$ become infinitesimals of sufficiently high order when $x \to 0$ we find, that in this case the solution of (3.1) is still unstable.

Indeed, let us assume that $b_0 = \sigma_0^2/2$ and that

$$|b(t, x) - b_0 x| + |\sigma(t, x) - \sigma_0 x| \leqslant c\,|x|^{1+\alpha} \tag{3.2}$$

holds for some $c > 0$ and $\alpha > 0$.

Let us consider an auxilliary function $V(x) = \ln\ln(1/|x|)$. A simple check shows that in this case $V \to \infty$ as $x \to 0$ and $LV < 0$ in a sufficiently small neighborhood of the coordinate origin.

Instability of the system (3.1) when $b_0 = \sigma_0^2/2$ and (3.2) follows from the Theorem 2.3 of [8].

A question whether an extension of this result to the multi-dimensional case is true, is of interest. Another question arises in this connection: is it not enough to require in the statements of Theorems 2.3 and 2.4 that conditions (2.4) and (2.5) hold for any single value of x? We know that this is possible in the case of deterministic systems. Results of [9] imply that it is indeed possible also for stochastic systems provided that the linearized system has constant coefficients and that its diffusion matrix is nondegenerate in the sense that

$$\sum_{r=1}^{k} (\sigma_r x, \lambda)^2 > 0 \tag{3.3}$$

for all vectors x and λ which are not null vectors. If, on the other hand, the condition (3.3) is not fulfilled, then we can find an example showing that such an improvement cannot, generally speaking, take place. It is however highly probable, that such an improvement can be made if the condition (1.15) is replaced in the statements of the above Theorems with another condition of the type (3.2).

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